General relation for stationary probability density functions

J. Mi and R. A. Antonia
Department of Mechanical Engineering, University of Newcastle, New South Wales 2308, Australia
(Received 27 January 1995)

A linear relation between a normalized, time (t) dependent, statistically stationary quantity (z) and the normalized conditional expectation (r) of $\partial^2 z/\partial t^2$ allows r to generally satisfy two conditions subject to the stationarity requirement. Experimental data for both temperature and vorticity in several turbulent flows indicate that this relation appears universal. As a result, the exact expression derived by Pope and Ching [Phys. Fluids A **5**, 1529 (1993)] for the probability density function (PDF) of any stationary quantity should generally reduce to the simpler form obtained by Ching [Phys. Rev. Lett. **70**, 283 (1993)].

PACS number(s): 47.27.Nz, 02.50.Ey, 05.40.+j

Recently, Pope and Ching [1] obtained an exact expression for the probability density function (PDF) of any normalized fluctuating quantity z measured in a general stationary process, in terms of conditional expectations of its time derivatives. The expression is

$$p(z)=\frac{C}{q(z)}\exp\left[\int_0^z \frac{r(z')}{q(z')}dz'\right] . \tag{1}$$

Here,

$$z \equiv (Z-\langle Z\rangle)/\langle (Z-\langle Z\rangle)^2\rangle^{1/2} ,$$

where Z is the instantaneous quantity and $\langle\ \rangle$ denotes a time average. C is a constant determined by the condition $\int_{-\infty}^{\infty} p(z)dz=1$, and $q(z)$ and $r(z)$ are given by

$$q(z)=\frac{\langle z_{,t}^2|z\rangle}{\langle z_{,t}^2\rangle} \tag{2}$$

and

$$r(z)=\frac{\langle z_{,tt}|z\rangle}{\langle z_{,t}^2\rangle} , \tag{3}$$

respectively. In (2) and (3), $z_{,t}\equiv\partial z/\partial t$ and $z_{,tt}\equiv\partial^2 z/\partial t^2$ and $\langle Q|z\rangle$ ($Q\equiv z_{,t}^2$ or $z_{,tt}$) is the expectation of Q conditioned on a particular value of z. Equation (1) imposes only two (weak) conditions: $z(t)$ is twice differentiable and $p(z)$ decreases sufficiently rapidly as $|z|\to\infty$. Obviously, these conditions are generally satisfied by most turbulent quantities (e.g., velocity, temperature, concentration, mass fraction, vorticity).

Prior to the derivation of Eq. (1), Ching [2] obtained, on the basis of the work by Sinai and Yakhot [3],

$$p(z)=\frac{C}{q(z)}\exp\left[-\int_0^z \frac{z'}{q(z')}dz'\right] \tag{4}$$

for both the temperature fluctuation θ and its time difference $\Delta\theta$ [$\equiv\theta(t+\tau)-\theta(t)$], by assuming $\langle z^{2n}\rangle=(2n-1)\langle z^{2n-2}y^2\rangle$ (where $z\equiv\beta/\langle\beta^2\rangle^{1/2}$ and $y\equiv\beta_{,t}/\langle\beta_{,t}^2\rangle^{1/2}$; β stands for either θ or $\Delta\theta$). Equation (4) is identical to Eq. (1) if

$$r(z)=-z . \tag{5}$$

Using the convective turbulence temperature data of Heslot, Castaing, and Libchaber [4], Ching found that (4) works well for both θ and $\Delta\theta$ (when τ is large). Pope and Ching [1] later confirmed that this is due to Eq. (5) being valid for the turbulence data. In the present paper, Eq. (5) is shown to be the solution for $r(z)$ which generally satisfies two conditions (see below) subject to z satisfying stationarity. Experiments in several (stationary) tur-

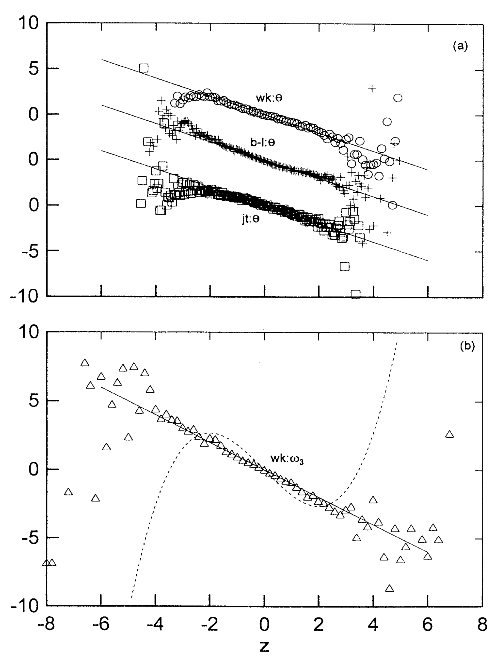

FIG. 1. Verification of Eq. (5) using (a) temperature and (b) spanwise vorticity fluctuations. Cylinder wake (40 diameters downstream of the cylinder and on the centerline): ○, $z\equiv\theta/\langle\theta^2\rangle^{1/2}$; ∇, $\omega_3/\langle\omega_3^2\rangle^{1/2}$. Boundary layer (0.012δ from the wall, where δ is the boundary layer thickness): +, $\theta/\langle\theta^2\rangle^{1/2}$. Round jet (30 nozzle diameters downstream of the jet exit and on the axis): □, $\theta/\langle\theta^2\rangle^{1/2}$. ——, Eq. (5); – – –, Eq. (14) with $n=1$ and $a=\sqrt{2}$.

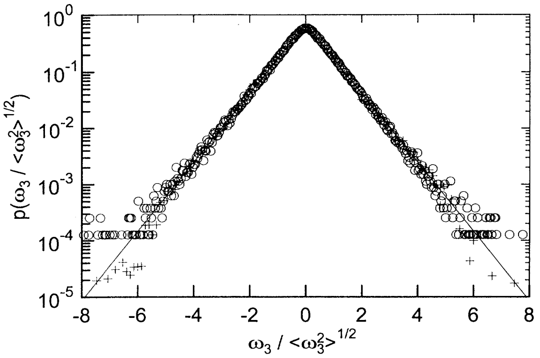

FIG. 2. The PDF of $z \equiv \omega_3/\langle \omega_3^2 \rangle^{1/2}$ on the wake centerline. ○, measurement; +, calculation. ——, $p(z)=(1/\sqrt{2})\exp(-\sqrt{2}|z|)$.

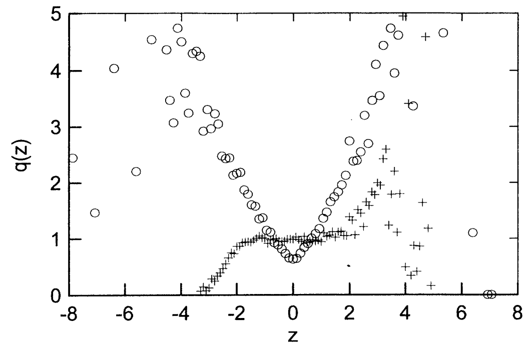

FIG. 3. Variation of q with z one the wake centerline. ○, $z \equiv \omega_3/\langle \omega_3^2 \rangle^{1/2}$; +, $\theta/\langle \theta^2 \rangle^{1/2}$.

bulent flows appear to point to the universality of this solution.

Differentiating z^2 twice with respect to time, we obtain

$$(z^2)_{,tt}=2[z_{,t}^2+zz_{,tt}] \ . \tag{6}$$

Averaging (6) with respect to time yields

$$\langle zz_{,tt} \rangle = -\langle z_{,t}^2 \rangle \tag{7}$$

since $\langle (z^2)_{,tt} \rangle = \langle z^2 \rangle_{,tt}=0$ for any stationary quantity. Using the definition of the conditional PDF (e.g., [5]), it can be shown that

$$\langle z \langle z_{,tt}|z \rangle \rangle = \langle zz_{,tt} \rangle \ . \tag{8}$$

It follows from (3), (7), and (8) that

$$\langle zr(z) \rangle = -1 \ . \tag{9}$$

In addition, the time average of $r(z)$ is zero, i.e.,

$$\langle r(z) \rangle = 0 \ , \tag{10}$$

since

$$\langle r(z) \rangle = \langle \langle z_{,tt}|z \rangle / \langle z_{,t}^2 \rangle \rangle = \langle z_{,t}^2 \rangle^{-1} \langle z_{,tt} \rangle$$

and $\langle z_{,tt} \rangle=0$. Equations (9) and (10) are identities for the general stationary quantity $z(t)$. Recalling that $\langle z^2 \rangle \equiv 1$ (the normalization condition) and

$$\langle F(z) \rangle \equiv \int_{-\infty}^{\infty} F(z)p(z)dz \ ,$$

where $F(z)$ is a function of z, (9) and (10) may be rewritten as

$$\int_{-\infty}^{\infty} z[r(z)+z]p(z)dz=0 \tag{11}$$

and

$$\int_{-\infty}^{\infty} r(z)p(z)dz=0 \ . \tag{12}$$

Obviously, (11) and (12) are both satisfied if

$$r(z)=-z \ ,$$

i.e., Eq. (5) is a mathematical solution for r. This solution allows r to generally satisfy (10) and (11), regardless of the particular form of $p(z)$. Therefore, if there exists a universal relation for r, it could be (5). As mentioned earlier, Pope and Ching [1] provided some experimental evidence in support of (5). To further test the validity of (5), we estimated r using temperature (θ) and spanwise vorticity (ω_3) data obtained [6] in three turbulent shear flows: a boundary layer over a rough wall, the wake of a circular cylinder, and a round jet. As shown in Fig. 1, Eq. (5) is quite well satisfied by both $z \equiv \theta/\langle \theta^2 \rangle^{1/2}$ and $z \equiv \omega_3/\langle \omega_3^2 \rangle^{1/2}$ in all three flows. The large scatter at large $|z|$ is associated with the small probability of occurrence of large values of $|z|$. We also estimated r when $z \equiv \Delta\theta/\langle (\Delta\theta)^2 \rangle^{1/2}$ for several values of τ; the results (not shown) were in close agreement with (5).

It is possible that there may be other mathematical solutions for r aside from (5), when $p(z)$ assumes some symmetrical forms. For example, if

$$p(z)=\frac{a}{2}\exp(-a|z|) \ , \quad a>0 \ , \tag{13}$$

(11) and (12) are valid when either $r(z)=-z$ or

$$r(z)=\frac{2a^2}{(2n+2)!}z^{2n+1}-z^{2n-1}-z \ , \tag{14}$$

where the integer n is greater than or equal to 1. Physically, however, the solution for r must be unique [this also applies to $q(z)$ and $p(z)$]. It follows that, even when $p(z)$ assumes the form described by (13), solutions (5) and (14) cannot both be valid. Figure 2 shows that the PDF of $z \equiv \omega_3/\langle \omega_3^2 \rangle^{1/2}$ on the wake centerline (40 diameters downstream of the cylinder) is adequately described by (13) with $a \approx \sqrt{2}$ [note that the calculation, based on Eq. (4), is in good agreement with measurement]. Yet the corresponding data for r closely follow (5) and not (14) [see Fig. 1(b)]. This, together with evidence presented in [1] and Fig. 1(a), and the observation that the statistical correlation between r and z is generally described by (9), all point to the likely universality of (5). By contrast, however, the level of correlation between q and z (Fig. 3) shows that there is a significant difference in q for two choices of z in the wake. The nonuniversality of q would be consistent with the nonuniversality of p.

Pope and Ching [1] showed that Eq. (5) is not satisfied by the Lorenz model [7] and by convective turbulence data [4] for $\Delta\theta$ when τ is small. This is not surprising because the Lorenz model and the small τ data shown in Figs. 1(b) and 2(a) of [1] do not satisfy both (11) and (12). In general, the measured difference $(\Delta\theta)_m$ is noise contaminated, i.e., $(\Delta\theta)_m = \Delta\theta + n$, where $\Delta\theta$ is the true difference and n represents the noise contribution. As $\tau \rightarrow 0$, $\Delta\theta \rightarrow 0$ and n makes a major contribution to $(\Delta\theta)_m$ [6]. Such data of $(\Delta\theta)_m$ cannot therefore well satisfy both (11) and (12); as a consequence, Eq. (5) is more strongly violated near $\Delta\theta = 0$ when τ is smaller.

[1] S.B. Pope and E. S. C. Ching, Phys. Fluids A **5**, 1529 (1993).
[2] E. S. C. Ching, Phys. Rev. Lett. **70**, 283 (1993).
[3] Y. G. Sinai and V. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989).
[4] F. Heslot, B. Castaing, and A. Libchaber, Phys. Rev. A **36**, 5871 (1987).
[5] S. B. Pope, Prog. Energy Combust. Sci. **11**, 119 (1985).
[6] R. A. Antonia and J. Mi, J. Fluid Mech. **250**, 531 (1993); J. Mi and R. A. Antonia, Expt. Therm. Fluid Sci. **8**, 328 (1994); H. S. Shafi *et al.* (unpublished).
[7] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).